Swimming, Pumping and Gliding at low Reynolds numbers

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Abstract

Simple, linear equations relate microscopic swimmers to the corresponding gliders and pumps. They have the following set of consequences: The swimming velocity of free swimmers can be inferred from the force on the tethered swimmer and vice versa; A tethered swimmer dissipates more energy than a free swimmer; It is possible to swim with arbitrarily high efficiency, but it is impossible to pump with arbitrarily high efficiency and finally that pumping is geometric. We also solve several optimization problems associated with swimming and pumping: The problem of optimal anchoring for a certain class of swimmers that includes the Purcell swimmer and the three linked spheres and the optimal geometries of helices considered as swimmers and pumps.

1 Introduction

Low Reynolds numbers hydrodynamics governs the locomotion of tiny natural swimmers such as micro-organisms and tiny artificial swimmers, such as microbots. It also governs micro-pumps [1], tethered swimmers [2, 3] and gliding under the action of external forces. Our purpose here is to discuss some interesting consequences of simple, yet general and exact relations that relate pumping, swimming and gliding at low Reynolds numbers. The equations are not new, and appear for example in [4, 5], where they serve to

analyze swimming at low Reynolds numbers. The interpretation of these equations as a relation between three apparently different objects: Swimmers, pumps and gliders, gives them a new and different flavor and suggests new applications which we explore below.

Our first tool is Eq. (1) which relates the velocity of a free swimmer to the (torque and) force needed to anchor it to a fixed location. This equation appears in the analysis of swimmers [4, 5]. It can be applied to experiments on tethered swimmers [6, 7, 8] where it can be used to infer the velocity of a free swimmers from measurements of the force on tethered one (and vice versa) and when both are measured, to an estimate of the gliding resistance matrix M of Eq. (3).

Our second tool is Eq. (7) which says that the power needed to operate a pump — a tethered swimmer — is the sum of the power invested by the corresponding free swimmer, and the power needed to tow it. (The result holds for arbitrary container of any shape or size). It is useful in studying and comparing the efficiencies of swimmers and pumps. In particular, it says that a tethered swimmer spends *more power* than a free swimmer. As we shall see it also implies that it is impossible to pump with arbitrary high efficiency.

Physically, both relations may be viewed as an expression of the elementary observation that swimmers and pumps are the flip sides of the same object: A tethered swimmer is a pump [2] and an unbolted pump swims. This observation plays limited role in high Reynolds numbers, because it does not lead to any useful equations for the non-linear Navier Stokes equations. For microscopic objects, where the equations reduce to the linear Stokes equations, the observation translates to linear relations.

Formally, a pump is a swimmer with a distinguished point — the point of anchoring. The pump may have different properties depending on the point of anchoring, (we shall see an example where it pumps in different directions depending on the anchoring point). We shall solve a problem, posed by E. Yariv [4], of how to find the optimal anchoring point for a certain class of swimmers, which we call "linear swimmers". The class includes the Purcel swimmer [9], the "three linked spheres" [10] and pushmepullyou [11].

There are various notions of optimality that one needs to consider in the study of swimmers and pumps. For swimmers, one notion of optimality is to maximize the distance covered in one stroke. A different notion is to minimize the dissipation in covering a given distance at a given speed. Similarly, for a pump one may want to maximize the momentum transfer to

the fluid in one stroke, or, alternatively, to minimize the dissipation for given total momentum transfer and rate. How are optimal swimmers related to optimal pumps? We study this question for helices and show that optimal swimmers and pumps have different geometry: There are four optimal pitch angles depending on what one optimizes and whether the helix is viewed as a swimmer or a pump.

2 Anchoring as a choice of gauge

Let us start by briefly discussing the gauge issues that arise when considering swimming [12], pumping and gliding.

Fixing a gauge is already an issue for a glider. A glider is a rigid body undergoing an Euclidean motion under the action of an external force. There is no canonical way to decompose a general Euclidean motion into a translation and a rotation [13]. Such a decomposition requires choosing a fiducial reference point in the body to fix the translation. In the theory of rigid body a natural choice is the center of mass. This is, however, *not* a natural choice at low Reynolds number. This is because this is the limit when inertia plays no role and the glider may be viewed as massless. Since the choice becomes arbitrary, it may be viewed as a choice of gauge.

In the case of a swimmer one needs to fix a point arbitrarily and in addition, to fix a fiducial frame. This is because a swimmer is a deformable body and such a frame is required to fix the rotation [12]. This is, again, a choice of gauge.

A pump is an anchored swimmer with a distinguished point and a distinguished frame which are determined by the way the pump is anchored. When we write equation that involve swimmers, pumps and gliders, we pick the fiducial point and frame determine by the the way the pump is anchored.

3 Triality of swimmer pumps and gliders

We shall first recall the derivation of the linear relation between the 6 dimensional force-torque vector, $\mathbf{F}_p = (F_p, N_p)$ which keeps a *pump* anchored with fixed position and orientation, and the 6 dimensional velocity–angular-velocity vector, $\mathbf{V}_s = (V_s, \omega_s)$ associated with the corresponding *autonomous*

swimmer:

$$\mathbf{F}_p = -M\,\mathbf{V}_s, \tag{1}$$

This holds for all times ($\mathbf{F}_p, M$ and $\mathbf{V}_s$ are time dependent quantities). M is a 6×6 matrix of linear-transport coefficients of the corresponding *glider*:

$$\mathbf{F}_g = M\,\mathbf{V}_g \tag{2}$$

namely, the corresponding rigid body, moving at (generalized) velocity $\mathbf{V}_g$ under the action of the (generalized) force $\mathbf{F}_g$. Note the change in sign. The matrix M depends on the geometry of the body. It is a positive matrix of the form [14]:

$$M = \begin{pmatrix} K & C \\ C^t & \Omega \end{pmatrix} \tag{3}$$

where K, C, and Ω are 3×3 real matrices. Note that Eq. (2) fails in two dimensions [15].

Eq. (1) follows from the linearity of the Stokes equations and the no-slip boundary conditions. Let $\partial\Sigma$ denote the surface of the device. Any vector field on $\partial\Sigma$ can be decomposed into a deformation and a rigid body motion as follows: Any rigid motion is of the form $\mathbf{v}_g = V + \omega \times \mathbf{x}$. Pick V to be the velocity of a fiducial point and ω the rotation of the fiducial frame. The deformation field is then, by definition, what remains when the rigid motion is subtracted from the given field $\mathbf{v}$.

Now, decompose $\mathbf{v}_s$ the velocity field on the surface of a swimmer, to a deformation and rigid-motion as above. The deformation field can be identified with the velocity field at the surface of the corresponding pump $\mathbf{v}_p$ since the pump is anchored with the fiducial frame that neither moves nor rotates. The remaining rigid motion $\mathbf{v}_g$ is then naturally identified with the velocity field on the surface of the glider. The three vector fields are then related by

$$\mathbf{v}_s = \mathbf{v}_p + \mathbf{v}_g, \quad \mathbf{v}_g = V_s + \omega_s \times \mathbf{x} \tag{4}$$

where V_s is (by definition) the swimming velocity and ω_s the velocity of rotation.

Each of the three velocity fields on $\partial\Sigma$, (plus the no-slip zero boundary conditions on the surface of the container, if there is one), uniquely determine the corresponding velocity field and pressure (v, p) throughout the fluid. The stress tensor, π_{ij}, depends linearly on (v, p) [15] .

By the linearity of the Stokes, $\partial_j \pi_{ij} = 0$, and incompressibility $\partial_j v_j = 0$ equations, it is clear that $v_s = v_g + v_p$ and $p_s = p_g + p_p$ and then also $\pi_s = \pi_p + \pi_g$. Since $F_i = \int_{\partial\Sigma} \pi_{ij} dS_j$ is the drag force acting on the device we get that the three force vectors are also linearly related: $F_s = F_p + F_g$, and similarly for the torques. This is summarized by the force-torque identity $\mathbf{F}_s = \mathbf{F}_p + \mathbf{F}_g$. Since the force and torque on an autonomous Stokes swimmer vanish, Eq. (1) follows from Eq. (2).

Eq. (1) has the following consequences:

- Micro-Pumping and Micro-Stirring is geometric: The momentum and angular momentum transfer in a cycle of a pump, $\int \mathbf{F}_p dt$, is independent of its (time) parametrization. In particular, it is independent of how fast the pump runs. This is because swimming is geometric [9, 12] and the matrix M is a function of the pumping cycle, but not of its parametrization.

- Scallop theorem for pumps: One can not swim at low Reynolds numbers with self-retracing strokes. This is known as the "Scallop theorem" [9]. An analog for pumps states that there is neither momentum nor angular momentum transfer in a pumping cycle that is self-retracing. This can be seen from the fact that $\mathbf{V}_s\, dt$ is balanced by $-\mathbf{V}_s\, dt$ when the path is retraced, and this remains true for $M\mathbf{V}_s\, dt$.

We shall now derive an equation, originally due to [5], which relates the power expenditure of swimmers, pumps and gliders. It follows from Lorentz reciprocity for Stokes flows that says that if (v_j, π_{jk}) and (v'_j, π'_{jk}) are the velocity and stress fields for two solutions of the Stokes equations in the domain Σ then [14]:

$$\int_{\partial\Sigma} v'_i\, \pi_{ij}\, dS_j = \int_{\partial\Sigma} v_i\, \pi'_{ij}\, dS_j \tag{5}$$

For the problem at hand, we may take $\partial\Sigma$ to be the surface of our device (since the velocity fields vanish on the rest of the boundary associated with the container). The area element dS is chosen normal to the surface and pointing into the fluid. Now apply the Lorentz reciprocity to a pump and a swimmer velocity fields and use Eq. (4) on both sides. This gives

$$-P_s + \mathbf{V}_s \cdot \mathbf{F}_s = -P_p - \mathbf{V}_s \cdot \mathbf{F}_p \tag{6}$$

where P_s is the power invested by the swimmer and P_p the power invested by the pump. Since the force and torque on the swimmer vanish, $\mathbf{F}_s = 0$, we get, using Eq. (1) a linear relation between the powers:

$$P_p - P_s = -\mathbf{V}_s \cdot \mathbf{F}_p = \mathbf{V}_s \cdot M\mathbf{V}_s = P_g \geq 0 \tag{7}$$

P_g is the power needed to tow the glider. Since both swimming and towing require positive power, at any moment pumping is more costly than swimming or dragging.

The linearity of Eq. (7) is a noteworthy, and somewhat unexpected. Eq. (4) says that the corresponding velocity fields are linearly related. Since power at low Reynolds number is quadratic in the velocity a linear relation between the powers is not what one may naively expect.

The most interesting consequence of this relation which, at least for us, was somewhat of a surprise, is that a pumps needs more power than a swimmer and so the power consumed by a tethered swimmer is actually an upper bound on the power consumed by it when freed.

One of the remarkable facts about low Reynolds number swimming is that even though the dynamics is governed by dissipation it is still possible to swim with arbitrarily high efficiency [11, 16]. For this to happen, the swimming velocity should be non-zero, but the energy dissipation of the swimmer, P_s, should be zero, thus $P_p = P_g \geq 0$. The same cannot happened for pumps: for a pump to be with arbitrarily hight efficiency, it must have non-zero momentum transfer to the fluid and zero power. Since M of Eq. (3) is a strictly positive matrix P_g is quadratic in the force and can not vanish if the pump transfers momentum to the fluid.

4 Linear swimmers and optimal anchoring

E. Yariv [4] posed the following problem: Find the anchoring point which optimizes the momentum transfer to the fluid. The general case is complicated. A class of swimmer for which this question can be answered relatively easily is the class of "linear swimmers" — swimmers made of segments, so that the velocity of different points on the same segment depend linearly on the distance between the points. This class contains the three linked spheres [10], the pushmepullyou [11], Purcell's three linked swimmer [17, 18], the 'N-Linked' swimmer [19], but not, for example, the treadmiller [16].

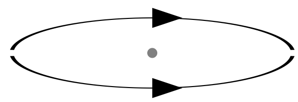

Figure 1: A treadmiller transports its skin from head to back. The treadmiller in the figure swims to the left while the motion of its skin guarantees that the ambient fluid is left almost undisturbed. If it is anchored at the gray point then it transfers momentum to the fluid (to the right). The power of towing a frozen treadmiller and of pumping almost coincide reflecting the fact that a treadmiller swims with little dissipation.

Shifting the anchoring point by $\vec{r}$, the resistance matrixes K, C and Ω of Eq. 3 will be changed to $K_r = K$, $C_r = C - KR$ and $\Omega_r = \Omega - RKR + CR - RC^t$ [14], where R is the antisymmetric matrix associated with the vector $\vec{r}$. The change in V_s is linear in $\vec{r}$, and ω_s is unchanged [13]. Using Eq. (1), the change in the force F_p (and hence in the linear-momentum transfer to the fluid) is linear with $\vec{r}$, and the optimal point which maximized the force must be at one of the edges of the segments.

(The change in the torque T_p (and hence in the angular-momentum transfer to the fluid) is quadratic in $\vec{r}$, so the maximum can be either at the edges, or at the point in which the differential of the angular momentum with respect to the anchoring point is zero (in cases where there is such a point). In either case, one has to check only a few points.)

A case in point is the “three linked spheres” [10]. There are three candidates for the optimal anchoring point: The three spheres. The two outer spheres are related by symmetry so the two interesting cases are either anchoring on an outer sphere or in the middle sphere. Detailed calculations show [20] that the maximum momentum transfer will be when the anchoring point is on one of the outer spheres. When the swimmer is anchored on the middle sphere, the momentum transfer turns out to be *in the opposite direction*. Calculation for the efficiency shows that the optimal point in this case is also at the outer spheres.

5 Helices as swimmers and pumps

Eq. (1) and Eq. (7) have an analogs for non-autonomous rigid swimmers such as a helix rotating by the action of an external torque [21]. This is a case that is very easy to treat separately. From Eq. (2) applied to the helix twice, once as swimmer and once as a pump we get the analog of Eq. (1):

$$F_p = C\omega = -KV_s \tag{8}$$

The analog of Eq. (7) follows immediately from the definition of the power $P = -\mathbf{F} \cdot \mathbf{V}$ and Eq. (2) again:

$$P_p - P_s = \mathbf{F}_p \cdot \mathbf{V}_p - \mathbf{F}_s \cdot \mathbf{V}_s = -V_s \cdot F_p = P_g \tag{9}$$

It follows from this that the difference in power between a swimmer and a pump is minimized, for given swimming velocity, if the swimming direction coincides with the smallest eigenvalue of K which is the direction of optimal gliding.

The distinction between optimal pumps and optimal swimmers [22] can be nicely illustrated by the considering the example of a rotating helix, or a screw. This motion has been studied carefully in [23, 24] whose analysis goes beyond what we need here. For a thin helix the slender-body theory of Cox [25] disposes of much of the hard work. Cox theory has the small parameter $(\log \kappa)^{-1}$ where κ is typically the ratio of the (large) radius of curvature of the slender body, r in the case of a helix, to its (small) diameter. To leading order in $(\log \kappa)^{-1}$ the local force field on the body is fixed by the *local* velocity field:

$$dF(x) = k(\mathbf{t}(\mathbf{t} \cdot \mathbf{v}) - 2\mathbf{v})dx, \quad k = \frac{2\pi\mu}{\ln \kappa} \tag{10}$$

$\mathbf{t}(x)$ is a unit tangent vector to the slender-body at x and $\mathbf{v}(x)$ the velocity of the point x of the body. This result may be interpreted as the statement that each line element has twice the resistance in the transverse direction than the resistance in the longitudinal direction, and that the motion of one line element does not affect the force on another element (to leading order).

Consider a helix of radius r, pitch angle θ and total length ℓ. The helix is described by the parameterized curve

$$(r\cos\phi, r\sin\phi, t\sin\theta), \quad \phi = \frac{t}{r}\cos\theta, \quad t \in [0, \ell] \tag{11}$$

Suppose the helix is being rotated at frequency ω about its axis. Substituting the velocity field of a rotating helix, with an unknown swimming velocity in the z-direction, into Eq. (10), and setting the total force in the z-direction to zero, fixes the swimming velocity. Dotting the force with the velocity and integrating gives the power. This slightly tedious calculation gives for the swimming velocity (along the axis) and the power of swimming:

$$\frac{V_s}{\omega r} = \frac{\sin 2\theta}{3+\cos 2\theta}, \quad \frac{P_s}{k\ell\omega^2 r^2} = \frac{4}{3+\cos 2\theta} \tag{12}$$

Similarly, for the pumping force and power one finds

$$\frac{F_p}{k\ell\omega r} = \sin\theta\cos\theta, \quad \frac{P_p}{k\ell\omega^2 r^2} = 1+\sin^2\theta \tag{13}$$

Eq. (12) and (13) have the following consequences for optimizing pumps and swimmers:

- Given ωr, a swimmer velocity V_s is maximized at pitch angle $\theta = 54.74°$.
- Given ωr, the pumping force F_p is maximized at $\theta = 45°$.

Consider now optimizing *both* the pitch angle θ and rotation frequency ω so that the swimming velocity is maximized for a given power. Namely

$$\max_{\theta,\omega}\{V_s \mid P_s = const\} \tag{14}$$

and similarly for pumping, except that F_p replaces V_s and P_p replaces P_s. A simple calculation shows that this is equivalent to optimizing V_s^2/P_s and F_p^2/P_p with respect to θ. (These ratios are independent of ω and so invariant under scaling time). One then finds:

- The efficiency of swimming, V_s^2/P_s, is optimized at $\theta = 49.9°$. The efficiency is proportional to $(k\ell)^{-1}$ which favors small swimmers in less viscous media, as one physically expects.
- The efficiency of pumping, F_p^2/P_p, is optimized at $\theta = 42.9°$. The efficiency is proportional to $(k\ell)$ which favors big pumps at more viscous media. Micro-pumps are perforce inefficient.

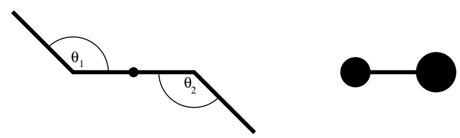

Figure 2: A Purcell three linked swimmer, left, controls the two angles θ_1 and θ_2. When bolted it effectively splits into two independent wind-shield wipers each of which is self retracing. It will, therefore, not pump. Pushmepullyou, right, controls the distance between the two spheres and the ratio of their volumes. It can have arbitrarily large swimming efficiency.

There is a somewhat unrelated, yet insightful fact that one learns from the above computation regarding the difference between motion in a very viscous fluid and motion in a solid. The naive intuition that the two are similar at very high viscosity would imply that a helix moves like a cork-screw and so would move one pitch in one turn. This is actually never the case, no matter how large μ is. In fact, the ratio of velocities of a helix to a cork-screw is independent of μ and by Eq. (12)

$$\frac{V_s}{\omega r \sin\theta} = \frac{\cos\theta}{1+\cos^2\theta} \leq \tfrac{1}{2} \tag{15}$$

A helix needs at least two turns to advance the distance of its threads.

6 Pumps that do no swim

We have noted that the "three linked spheres" can pump either to the right or to the left depending on whether it is anchored on the center sphere or on an external sphere. There is, therefore, an intermediate point where it will not pump.

There are also pumps that will not swim: Evidently, if the swimming stroke is right-left and up-down symmetric, the swimmer will not move by symmetry. It can, however, be bolted in a way that breaks the symmetry to give an effective pump. For example - consider a right-left symmetric pushmepullyou [11], where the two spheres inflate and deflate in phase. By symmetry, it will not swim, however, anchoring in any point except the middle point will lead to net momentum transfer.

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